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Model for Turbulent Transfer and Turbulence Dynamics in a Stratified Shear Flow

L. A. OSTROVSKIY AND YU. I. TROITSKAYA

Institute for Applied Physics, USSR Academy of Sciences

A kinetic approach has been used to obtain equations for turbulent flows for momentum, mass, energy and other hydrodynamic quantities in a stratified medium whose type is uniquely determined by the assumption that the probability distribution function is close to Gaussian. The behavior of turbulence is discussed in a stratified shear flow, and also in the field of an inertial-gravity internal wave. It is shown that taking into account the mutual transformation of the kinetic and potential fluctuation energy makes it possible to explain the possibility of maintaining turbulence in weak shear flows typical of internal waves in the main part of the ocean.

Small-scale motions occur practically everywhere in the ocean and in the atmosphere. Despite the relatively small amount of energy involved, they play an important role in energy transfer for geophysical processes, in particular providing a source of energy for flows and wave motions on larger scales.

The approach that has been most widely used in describing the interaction between turbulence and large-scale motions is associated with the use of a system of Reynolds equations [1, 2]. Because of the nonlinearity in the hydrodynamics equations this system is open: the equation for any moment contains a higher moment. It is therefore necessary to bring in a closure hypothesis; the ones usually used are gradient approximations for momentum flows, buoyancy and other quantities introduced in analogy with molecular transport transfer theory [2]. Although in many cases this approach turns out to be effective, it has obvious shortcomings. In particular, these approximations are introduced independently for different physical quantities that are related, generally speaking, to the equations of hydrodynamics. This in turn leads to a significant indeterminacy even in relation to the qualitative conclusions of the theory.

In the kinetic theory of gases there is a well-known mathematical series procedure for obtaining equations for the viscous stresses and molecular heat flows across velocity and temperature gradients. In this way the equation for the single-particle distribution function is solved [3]. An analogous approach has already been used for quite some time in the theory of hydrodynamic turbulence [4 - 6]. Here also there is the equation for the single-point distribution function f, the kinetic equation. Then, using only the one closure hypothesis in the equation for f, it is possible to obtain the equations consistent with each other for the flows for various hydrodynamic quantities. In order to do this it is necessary, having written the equation for f, to find an adequate approximation for the "collision integral" (here its role is played by terms related to pressure fluctuations and fluctuating viscous force components [4]), solve the resulting equation and, finally, knowing the distribution function, to calculate the Reynolds stresses using well-known equations from probability theory.

The present paper uses this kinetic approach to describe turbulence in a stratified fluid and its behavior in the field of current shear and internal waves.* In order to obtain here the general equations for the turbulent flows for momentum, density, energy and other hydrodynamic quantities it is not required, generally speaking, that any approximation for the collision integral be used. It is only required that the external turbulence scale be small in comparison with the average motion scale and that the distribution function differ slightly from a Gaussian one. However, the corresponding calculations are extremely cumbersome (see [18]). Here we restrict ourselves to a simpler approach related to the τ-approximation for the "collision integral," in which the term associated with the pressure fluctuations is the equation $-(f - f_0)/\tau$, where f_0 is a Gaussian function. This approximation was not

*We have already used this approach to describe the interaction between internal waves and the random current field [7] in a "collision-free" version, where the velocity field can be assumed "frozen" over the interaction time. Here a "strongly collisional" model is considered which adequately describes hydrodynamic turbulence.

validated in problems on hydrodynamic turbulence, and was introduced in [9] by analogy with the kinetic theory of rarefied gases. However, as has been shown in [8], it gives the same structure for the equations for the turbulent flows as also the more rigorous procedure, from which follow analogous results pertaining to the behavior of turbulence in a shear current field.

1. Kinetic Equation for a Stratified Turbulent Flow

We will proceed from the following form for the equations of hydrodynamics for a stratified fluid:

$$\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u},\nabla)\mathbf{u}+\frac{1}{\rho_0}\frac{\partial p}{\partial \mathbf{r}}+\mathbf{g}\frac{\rho-\rho_0}{\rho_0}=\nu\Delta\mathbf{u},$$
$$\frac{\partial \rho}{\partial t}+(\mathbf{u},\nabla)\rho=\varkappa\Delta\rho,\ \operatorname{div}\mathbf{u}=0, \tag{1}$$

where $\rho_0(z)$ is the hydrostatic density distribution, ν and $\varkappa$ are the molecular viscosity and molecular density diffusion coefficients, respectively, and p is the deviation of the pressure from hydrostatic pressure. In this case the stratification is assumed to be quite weak in order for the Boussinesq approximation to be valid; in addition, for simplicity, we are restricting ourselves to the equation for the total density balance (without separately including the temperature and salinity balance).

We introduce the distribution function as the probability density for the velocity and density distribution at a given point at a given time. By definition

$$f(\mathbf{v},\lambda,\mathbf{r},t)=\langle\delta(\mathbf{u}(\mathbf{r},t)-\mathbf{v})\,\delta(\rho(\mathbf{r},t)-\lambda)\rangle, \tag{2}$$

δ is the Dirac function, and the angular brackets denote probability averaging. Differentiating (2) with respect to t, substituting the expressions for $\partial u/\partial t$ and $\partial\rho/\partial t$ from the system (1) and making simple transformations, we obtain (as yet open) an equation for f

$$\frac{\partial f}{\partial t}+\mathbf{v}\frac{\partial f}{\partial \mathbf{r}}+\left(-\frac{1}{\rho_0}\frac{\partial\langle p\rangle}{\partial \mathbf{r}}-\frac{(\lambda-\rho_0)}{\rho_0}\mathbf{g}\right)\frac{\partial f}{\partial \mathbf{v}}=$$
$$=\frac{\partial}{\partial \mathbf{v}}\left\langle\delta(\mathbf{u}-\mathbf{v})\,\delta(\rho-\lambda)\left(\frac{1}{\rho_0}\frac{\partial p'}{\partial \mathbf{r}}-\nu\Delta\mathbf{u}'\right)\right\rangle+ \tag{3}$$
$$+\frac{\partial}{\partial\lambda}\langle\delta(\mathbf{u}-\mathbf{v})\,\delta(\rho-\lambda)(-\varkappa\Delta\rho')\rangle.$$

Here $p'=p-\langle p\rangle$, $\mathbf{u}'=\mathbf{u}-\langle\mathbf{u}\rangle$ and $\rho'=\rho-\langle\rho\rangle$ are the fluctuation components for the pressure, velocity and density. The pressure fluctuations are described by a Poisson equation easily following from the initial system (1):

$$\Delta p'=-\rho_0[\operatorname{div}(\mathbf{u},\nabla)\mathbf{u}]'-g\frac{\partial\rho'}{\partial z},$$

whose solution has the form

$$p'=p_1'+p_2'=\frac{1}{4\pi}\rho_0\int d\mathbf{r}_1\frac{1}{|\mathbf{r}-\mathbf{r}_1|}\operatorname{div}[(\mathbf{u}(\mathbf{r}_1,t),\nabla)\mathbf{u}(\mathbf{r}_1,t)]'+$$
$$+\frac{g}{4\pi}\int d\mathbf{r}_1\frac{1}{|\mathbf{r}-\mathbf{r}_1|}\frac{\partial}{\partial z_1}\rho'(\mathbf{r}_1,t). \tag{4}$$

Here $[\operatorname{div}(\mathbf{u},\nabla)\mathbf{u}]'=[\operatorname{div}(\mathbf{u},\nabla)\mathbf{u}]-\langle[\operatorname{div}(\mathbf{u},\nabla)\mathbf{u}]\rangle$. It is seen that Eq. (4) breaks down into two terms having different physical meaning. The first term represents pressure fluctuations due to the chaotic motion of the particles of the fluid, and the second term represents their random displacement from the equilibrium level in the stratified fluid. The corresponding terms in Eq. (3) are expressed through the two-point distribution function:

$$J_p=\frac{\partial}{\partial\mathbf{v}}\left\langle\delta(\mathbf{u}-\mathbf{v})\,\delta(\rho-\lambda)\frac{\partial p_1'}{\partial\mathbf{r}}\frac{1}{\rho_0}\right\rangle=-\frac{1}{4\pi}\frac{\partial}{\partial\mathbf{v}}\int d\mathbf{r}_1\times$$
$$\times\frac{\partial}{\partial\mathbf{r}}\frac{1}{|\mathbf{r}-\mathbf{r}_1|}\int d\mathbf{v}_1\left(\mathbf{v}_1\frac{\partial}{\partial\mathbf{r}_1}\right)^2[f_{2v}-ff_v], \tag{5}$$

$$J_\rho=\frac{\partial}{\partial\mathbf{v}}\left\langle\delta(\mathbf{u}-\mathbf{v})\,\delta(\rho-\lambda)\frac{\partial p_2'}{\partial\mathbf{r}}\frac{1}{\rho_0}\right\rangle=$$
$$=-\frac{g}{4\pi\rho_0}\int d\mathbf{r}_1\frac{\partial}{\partial\mathbf{r}}\frac{1}{|\mathbf{r}-\mathbf{r}_1|}\frac{\partial}{\partial z_1}\int_{-\infty}^{\infty}(\lambda_1-\rho(\mathbf{r}_1,t))f_{2\lambda}d\lambda_1. \tag{6}$$

The terms in (3) that describe the effect of the viscosity and molecular diffusion on the single-point distribution function f are also expressed through the two-point function:

$$J_\nu=\frac{\partial}{\partial\mathbf{v}}\langle-\nu\Delta\mathbf{u}'\delta(\mathbf{u}-\mathbf{v})\,\delta(\rho-\lambda)\rangle=$$
$$=-\nu\frac{\partial}{\partial\mathbf{v}}\lim_{\mathbf{r}_1\to\mathbf{r}}\frac{\partial}{\partial\mathbf{r}_1}\frac{\partial}{\partial\mathbf{r}_1}\iiint_{V_\infty}\mathbf{v}_1[f_{2v}-ff_v]\,d\mathbf{v}_1, \tag{7}$$

$$J_\varkappa=\frac{\partial}{\partial\lambda}\langle-\varkappa\Delta\rho'\delta(\mathbf{u}-\mathbf{v})\,\delta(\rho-\lambda)\rangle=$$
$$=-\varkappa\frac{\partial}{\partial\lambda}\lim_{\mathbf{r}_1\to\mathbf{r}}\frac{\partial}{\partial\mathbf{r}_1}\frac{\partial}{\partial\mathbf{r}_1}\int_{-\infty}^{\infty}\lambda_1[f_{2\lambda}-ff_\lambda]\,d\lambda_1. \tag{8}$$

In Eqs. (5) - (8) we obtain the distribution functions, the two-point distribution function with respect to velocity

$$f_{2v}(\mathbf{v},\mathbf{v}_1,\lambda,\mathbf{r},\mathbf{r}_1,t)=\langle\delta(\mathbf{u}-\mathbf{v})\,\delta(\mathbf{u}_1-\mathbf{v}_1)\,\delta(\rho-\lambda)\rangle,$$

and the two-point distribution function with respect to density

$$f_{2\lambda}(\mathbf{v},\lambda,\lambda_1,r,\mathbf{r}_1,t)=\langle\delta(\mathbf{u}-\mathbf{v})\,\delta(\rho-\lambda)\,\delta(\rho_1-\lambda_1)\rangle.$$

In addition, we have the notation

$$f_v(\mathbf{v}_1,\mathbf{r}_1,t)=\langle\delta(\mathbf{u}(\mathbf{r}_1,t)-\mathbf{v}_1)\rangle,\quad f_\lambda(\lambda_1,\mathbf{r}_1,t)=\langle\delta(\rho(\mathbf{r}_1,t)-\lambda_1)\rangle.$$

The derivation of Eqs. (5) - (8) has been given in [8]. The terms J_p, J_ρ, J_ν, and $J_\varkappa$ are analogous to the collision integrals in the kinetic theory of gases. Straightforward estimates show that $J_\rho\sim f/\tau_\rho$, where $\tau_\rho=\rho_0U/g\langle\rho'^2\rangle^{1/2}$, U^2 is the velocity dispersion, $\langle\rho'^2\rangle$ is the density dispersion, $J_\varkappa$, and $J_\nu\sim f/\tau_\nu$, where $\tau_\nu\sim U^2/\varepsilon$ and ε is the turbulent energy dissipation rate. The terms on the left side of the kinetic equation (3) are on the order of f/T and Uf/l, where T and l are the time and spatial scales for the mean motion. Finally, $J_p \sim f/\tau$, where $\tau \sim L/U$, with L being the turbulence scale.

Similar to [9], we use the τ-approximation for J_p:

$$J_p = -(f - f_0)/\tau.$$

Here

$$\tau = L/U,\ f_0 = \frac{1}{(2\pi U^2)^{3/2}} \frac{1}{(2\pi \langle \rho'^2 \rangle)^{1/2}} \exp\left\{-\frac{(\mathbf{v} - \langle \mathbf{u} \rangle)^2}{2U^2} - \frac{(\lambda - \langle \rho \rangle)^2}{2\langle \rho'^2 \rangle}\right\},$$

$$= \int \mathbf{v} f d\mathbf{v} d\lambda,\ \langle \rho \rangle = \int \lambda f d\mathbf{v} d\lambda$$

are the mean values for the velocity and density, and $U^2 = {}^1/_3 \int (v - \langle u \rangle)^2 f d\mathbf{v} d\lambda$, and $\langle \rho'^2 \rangle = \int (\lambda - \langle \rho \rangle)^2 f d\mathbf{v} d\lambda$ are respectively the dispersions in these quantities.

The equation for f then becomes

$$\frac{\partial f}{\partial t} + \mathbf{v}\frac{\partial f}{\partial \mathbf{r}} + \left(-\frac{1}{\rho_0}\frac{\partial \langle p \rangle}{\partial \mathbf{r}} - \frac{\lambda - \rho_0}{\rho_0}\mathbf{g}\right)\frac{\partial f}{\partial \mathbf{v}} = -\frac{f - f_0}{\tau} + J_\rho + J_\nu + J_\varkappa, \tag{9}$$

where J_ρ, J_ν, and $J_\varkappa$ are given by Eqs. (6) - (8).

We consider now those processes for which $\tau \ll \{\tau_\rho, \tau_\nu, T, l/U\}$. These inequalities are often characteristic of geophysical situations. They signify, as has already been noted, the small nature of the external turbulence scale as compared with the scales for the mean motions, and also with the shortness of the relaxation time for the distribution function for the turbulent flow to a Gaussian one in comparison with its characteristic variation times due to viscous damping, the operation of buoyancy forces, and transfer to the mean flow. We shall make some estimates of the applicability of these inequalities for typical ocean turbulence parameters [2]: L = 10 cm, U = 0.5 cm/sec, and $\langle \rho'^2 \rangle^{1/2} = 10^{-6}$ g/cm^3, so then τ = 50 sec and τ_ρ = 500 sec. The condition $\tau \ll \tau_\nu$ is automatically satisfied in developed turbulence. The time scale for the averaged processes should exceed 50 sec, and the spatial scale should be greater than 10 cm. These conditions are nearly always satisfied for internal waves in the ocean.

In this case all the terms in Eq. (9), except for $-(f - f_0)/\tau$ are on the order of the small parameter $\mu = \tau/\{\tau_\nu, \tau_\rho, T\}$ and it is possible to find a solution of (9) in the form of an expansion

$$f = f^{(0)} + f^{(1)} + \ldots \tag{10}$$

In the kinetic theory of gases this is the well-known Chapman-Enskog method [10]. In the expansion (10) the ratio of each succeeding term to the preceding one takes on values on the order of μ; $f^{(0)}$ satisfies the equation for the zero approximation, following from (9): $f^{(0)} = f_0$, i.e., the solution of the zero approximation is a Gaussian function. The solution to the equation for the first approximation has the form

$$f^{(1)} = -\tau\left\{\frac{\partial f_0}{\partial t} + \mathbf{v}\frac{\partial f_0}{\partial \mathbf{r}} + \left(-\frac{1}{\rho_0}\frac{\partial \langle p \rangle}{\partial r} - \frac{\lambda - \rho_0}{\rho_0}\mathbf{g}\right)\frac{\partial f_0}{\partial \mathbf{v}} - J_\rho[f_0, f_{02\lambda}] - J_\nu[f_0, f_{02\mathbf{v}}] - J_\varkappa[f_0, f_{02\lambda}]\right\}. \tag{11}$$

The integrals J_ρ, J_ν and $J_\varkappa$ in Eq. (11) contain the two-point Gaussian functions $f_{0\lambda}$ and $f_{0\mathbf{v}}$. Substituting them in (6) - (8), we have

$$\begin{aligned} J_\rho &= \frac{\partial}{\partial \mathbf{v}}\left(\frac{f_0(\lambda - \langle \rho \rangle)\beta}{\langle \rho'^2 \rangle}\right)\frac{\mathbf{g}}{\rho_0}, \\ \beta &= \frac{1}{4\pi}\int d\mathbf{r}_1 \frac{\partial}{\partial \mathbf{r}}\frac{1}{|\mathbf{r} - \mathbf{r}_1|}\frac{\partial}{\partial z_1}\langle \rho'(\mathbf{r}, t)\rho'(\mathbf{r}_1, t)\rangle, \\ J_\nu &= \frac{\varepsilon}{3U^2}\frac{\partial}{\partial \mathbf{v}}\left((\mathbf{v} - \langle \mathbf{u} \rangle)\frac{\partial f}{\partial \mathbf{v}}\right), \\ \varepsilon &= -\lim_{\mathbf{r}_1 \to \mathbf{r}} \nu \frac{\partial}{\partial \mathbf{r}_1}\frac{\partial}{\partial \mathbf{r}_1}\langle u_i'(\mathbf{r}, t) u_i'(\mathbf{r}_1, t)\rangle, \\ J_\varkappa &= \frac{\Gamma}{\langle \rho'^2 \rangle}\frac{\partial}{\partial \lambda}\left((\lambda - \langle \rho \rangle)\frac{\partial f}{\partial \lambda}\right), \\ \Gamma &= -\lim_{\mathbf{r}_1 \to \mathbf{r}} \varkappa \frac{\partial}{\partial \mathbf{r}_1}\frac{\partial}{\partial \mathbf{r}_1}\langle \rho'(\mathbf{r}, t)\rho'(\mathbf{r}_1, t)\rangle. \end{aligned} \tag{12}$$

Thus, $f^{(1)}$ is expressed in terms of f_0. Using the well-known distribution function it is possible to find an equation for all the single-point moments using standard equations from probability theory

$$\left\langle \rho'^{m_4}\prod_{i=1}^{3} u_i'^{m_i}\right\rangle = \int (f^{(0)} + f^{(1)})(\lambda - \langle \rho \rangle)^{m_4}\prod_{i=1}^{3}(v_i - \langle u_i \rangle)^{m_i} d\mathbf{v} d\lambda.$$

We write the equations for the second and third moments in explicit form. Thus, the turbulent momentum flow is

$$\langle u_i' u_j' \rangle = U^2\delta_{ij} - LU\left(\frac{\partial \langle u_i \rangle}{\partial x_j} + \frac{\partial \langle u_j \rangle}{\partial x_i}\right). \tag{13}$$

The equation for turbulent mass flow has the form

$$\langle \rho' u_i' \rangle = -LU\left(\frac{\partial \langle \rho \rangle}{\partial x_i} + g_i\frac{\langle \rho'^2 \rangle}{U^2\rho_0} - \frac{g\beta_i}{\rho_0 U^2}\right). \tag{14}$$

The equation for β is given by Eq. (12). For a statistically homogeneous density field $\beta_x = \beta_y = 0$, and $\beta_z = \langle \rho'^2 \rangle R$, where

$$R = -\frac{1}{4\pi}\int d\mathbf{r}_1 \frac{\partial}{\partial z}\frac{1}{|\mathbf{r} - \mathbf{r}_1|}\frac{\partial}{\partial z_1}B_\rho(\mathbf{r} - \mathbf{r}_1). \tag{15}$$

Here B_ρ is the correlation coefficient for the density field, R depends on the correspondence of

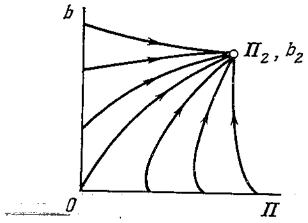

Fig. 1. Phase plane for the system (20).

the vertical L_z and horizontal L_r scales for density field correlation, and in fact $R \simeq 1$ for $L_z \ll L_r$, and $R \sim (L_r/L_z)^2$ for $L_z \gg L_r$.

In Eq. (14) there is an additional term $(g_i\langle\rho'^2\rangle - \beta_i g)/\rho_0 U^2$ for $\langle\rho' u_i'\rangle$ as compared with the gradient hypotheses. We compare it with the "gradient" term $\partial\langle\rho\rangle/\partial x_i$ for values for the parameters characteristic of the main water column for the ocean [2]: $\partial\langle\rho\rangle/\partial z = 9\cdot 10^{-7}$ g/cm$^3\cdot$m, [N = 3×10^{-3} sec^{-1}], and $\langle\rho'^2\rangle^{1/2} = 10^{-6}$ g/cm^3; U = 0.5 cm/sec; then the additional term will be of the same order ($\sim 4 \times 10^{-7}$ g/cm$^3\cdot$m) as the gradient term. From what follows it is seen that the presence of an additional term in the mass flow equation as compared with gradient hypotheses leads to a number of important differences in the results given by these models.

The equations for kinetic energy turbulent flows and for the density fluctuations have the ordinary "gradient" form:

$$\left\langle \sum_{j=1}^{3} u_j'^2 u_i' \right\rangle = -5LU\frac{\partial U^2}{\partial x_i}, \qquad (16)$$

$$\langle \rho'^2 u_i' \rangle = -LU\frac{\partial\langle\rho'^2\rangle}{\partial x_i}. \qquad (17)$$

The system of Reynolds equations, which is closed according to Eqs. (13) - (17), has the form

$$\frac{\partial\langle u_i\rangle}{\partial t} + \langle u_j\rangle\frac{\partial\langle u_i\rangle}{\partial x_j} + \frac{1}{\rho_0}\frac{\partial\langle p\rangle}{\partial x_i} + g_i\frac{\langle\rho\rangle - \rho_0}{\rho_0} = = \frac{\partial}{\partial x_j}\left(L\sqrt{b}\left(\frac{\partial\langle u_i\rangle}{\partial x_j} + \frac{\partial\langle u_j\rangle}{\partial x_i}\right)\right),$$

$$\frac{\partial\langle\rho\rangle}{\partial t} + \langle u_i\rangle\frac{\partial\langle\rho\rangle}{\partial x_i} = 2\frac{\partial}{\partial x_i}L\sqrt{b}\left(\frac{\partial\langle\rho\rangle}{\partial x_i} + \frac{3}{2b\rho_0}(g_i\langle\rho'^2\rangle - g\beta_i)\right),$$

$$\frac{\partial b}{\partial t} + \langle u_i\rangle\frac{\partial b}{\partial x_i} - L\sqrt{b}\left(\frac{\partial\langle u_i\rangle}{\partial x_j} + \frac{\partial\langle u_j\rangle}{\partial x_i}\right)^2 - \frac{g}{\rho_0}L\sqrt{b}\times \times\left(\frac{\partial\langle\rho\rangle}{\partial z} + \frac{3}{2b\rho_0}(g\langle\rho'^2\rangle - g\beta_z)\right) - \frac{Cb^{3/2}}{L} = \frac{5}{3}\frac{\partial}{\partial x_i}\left(L\sqrt{b}\frac{\partial b}{\partial x_i}\right),$$

$$\frac{\partial\langle\rho'^2\rangle}{\partial t} + \langle u_i\rangle\frac{\partial\langle\rho'^2\rangle}{\partial x_i} - 2\frac{\partial\langle\rho\rangle}{\partial x_i}L\sqrt{b}\times \times\left(\frac{\partial\langle\rho\rangle}{\partial x_i} + (g_i\langle\rho'^2\rangle - g\beta_i)\frac{3}{2b\rho_0}\right) = \frac{\partial}{\partial x_i}L\sqrt{b}\frac{\partial\langle\rho'^2\rangle}{\partial x_i}. \qquad (18)$$

Here $b = 3U^2/2$, and the summing is carried out using the repeating subscripts.

2. Evolution of Turbulence in the Field of a Stratified Shear Flow

As an example for the application of the developed theory we consider the problem of the evolution of homogeneous turbulence in which all the moments higher than the first are independent of the coordinates in a stably stratified steady shear flow with the velocity profile $\mathbf{u} = U_0(z)\mathbf{x}_0$, where U'_{0z} = const and a constant Brunt-Väisälä frequency $\left(N^2 = -\frac{g d\rho_0}{\rho_0 dz} = \text{const}\right)$. As is well known [1], the condition for the turbulence to increase in such a flow has the form

$$\mathrm{Ri} < 1/\varkappa_\rho, \qquad (19)$$

where $\mathrm{Ri} = N^2/U_{0z}^2$ and $\varkappa_\rho$ is the ratio of the turbulent transfer coefficient to the turbulent viscosity coefficient [1]. In the gradient semi-empirical hypotheses for closure $\varkappa_\rho$ is ordinarily assumed to be constant, and the condition (19) places a limitation on the growth of turbulence in terms of the Richardson number Ri. However, as field and laboratory experiments have shown [1], $\varkappa_\rho$ is a decreasing function of Ri. If this decrease is rather rapid, so that $\mathrm{Ri}\,\varkappa_\rho < 1$ everywhere, the condition (19) is satisfied for any Ri. The model developed here gives precisely such a decreasing relation for $\varkappa_\rho(\mathrm{Ri})$.

The turbulent energy balance is determined by the equations for the mean kinetic energy b and for the mean potential energy $\Pi = \langle\rho'^2\rangle g^2/2N^2\rho_0^2$:

$$\frac{db}{dt} = U_{0z}^2 L\sqrt{b} - N^2 L\sqrt{b}\left(1 - \frac{3\Pi}{b}(1-R)\right) - \frac{Cb^{3/2}}{L},$$
$$\frac{d\Pi}{dt} = N^2 L\sqrt{b}\left(1 - \frac{3\Pi}{b}(1-R)\right) - \frac{Db^{1/2}\Pi}{L}. \qquad (20)$$

We assume the external turbulence scale to be constant (L = const). This simple assumption is often made in solving problems on the evolution of turbulence in the ocean [2]. The parameter R is determined by Eq. (15). The Kolmogorov approximations $\varepsilon_t = Cb^{3/2}/L$ and $\Gamma_t = Db^{1/2}\Pi/L$ (where C and D are empirical constants) are taken for the turbulent energy dissipation rate $\varepsilon_t = \langle\nu(\partial u_i'/\partial x_j + \partial u_j'/\partial x_i)^2\rangle$ and for the diffusion rate for turbulent density fluctuations $\Gamma_t = \langle\varkappa(\partial\rho'/\partial x_i)^2\rangle$.

Under these conditions the phase plane for the system (20) has the form shown in Fig. 1. Independently of the value for Ri on the phase plane there are two equilibrium states. One of them ($\Pi_1 = b_1 = 0$) is an unstable equilibrium state of the saddle type, and the second (Π_2, b_2) is a stable state of a node type. All of the phase trajectories approach the point Π_2, b_2, i.e., an equilibrium state for the flow with nonzero fluctuation energy is established independently of the initial conditions. According to

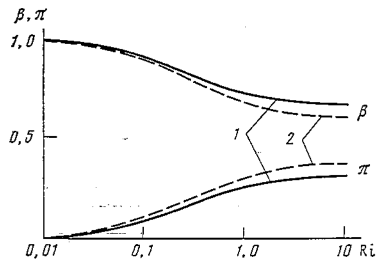

Fig. 2. Variation of the dimensionless kinetic $\beta=b/U_{0z}^2L^2$ and potential $\pi=\Pi/U_{0z}^2L$ energies of turbulence in a steady state as a function of Ri: 1) R = 0.3, 2) R = 0.5.

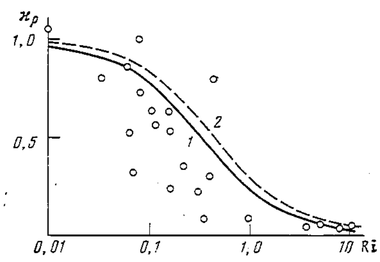

Fig. 3. Variation in $\varkappa_\rho(\mathrm{Ri})$ with R = 0.3 (1) and R = 0.5 (2). The experimental points have been taken from the composite diagram in [1].

[11], the numerical values for the constants appearing in (20) are $C = D = 0.09$. The equation for the turbulence energy in the steady state has the form

$$b_2=\frac{U_{0z}^2L^2}{2C}(1-\mathrm{Ri}(4-3R)+[(1-\mathrm{Ri}(4-3R))^2+12(1-R)\mathrm{Ri}]^{1/2}),$$

$$\Pi_2=\frac{U_{0z}^2L^2}{C}-b_2.$$

The variation of the steady-state values for the normalized kinetic and potential energies as a function of Ri is shown in Fig. 2. The characteristic time for establishing this steady state is $\sim 1/(U_{0z}\sqrt{C})$.

Let us consider in more detail the role of the additional term in the equation for the buoyancy flow (see Eq. (14)). As seen from (20), the gradient term $-N^2L\sqrt{b}$ in the mass flow equation describes the transfer of the energy from kinetic to potential. The additional term $\frac{3\Pi}{\sqrt{b}}(1-R)N^2$ has the opposite gradient sign and describes the transfer of energy from potential to kinetic due to the operation of buoyancy forces. If the Reynolds number is large ($\mathrm{Ri}\gg 1$), then for a sufficiently short time ($\sqrt{b}/(LN^2)$) the energy value is equalized for energy transferring from kinetic to potential energy and conversely, due to the operation of buoyancy forces. In this case, a correspondence $b=3\Pi(1-R)$ is established between the kinetic and potential turbulence energy, and the kinetic energy satisfies the equation

$$\frac{db}{dt}=L\sqrt{b}\frac{3(1-R)}{4-3R}U_{0z}^2-\frac{b^{3/2}(3(1-R)C+D)}{(4-3R)L}, \qquad (21)$$

which does not contain Ri. Thus, for large Ri the evolution of turbulence is independent of the Richardson number.

We calculate the turbulent Prandtl number for the steady-state case. By definition

$$\varkappa_\rho=\frac{\langle\rho'w'\rangle\, dU_0/dz}{\langle u'w'\rangle\, d\rho_0/dz}=1-\frac{3\Pi}{b}(1-R).$$

We have from the system (20)

$$\varkappa_\rho=(4-3R+1/\mathrm{Ri}-((4-3R)+1/\mathrm{Ri})^2-4/\mathrm{Ri})^{1/2})/2.$$

The function $\varkappa_\rho(\mathrm{Ri})$ is a decreasing one, and for $\mathrm{Ri}\gg 1$ it is

$$\varkappa_\rho(\mathrm{Ri})=1/[(4-3R)\mathrm{Ri}].$$

The condition for turbulence generation ($\mathrm{Ri}<1/\varkappa_\rho$) is satisfied for any Ri, i.e., there is no threshold for turbulence generation according to the Richardson number. The variation $\varkappa_\rho(\mathrm{Ri})$ is shown in Fig. 3. The experimental points have been taken from [1]; we see that the theory correctly describes the trend in the behavior of $\varkappa_\rho$ as a function of Ri.

3. Evolution of Turbulence in the Internal Wave Field

The variable velocity shear in an internal wave can also support small-scale turbulence. Suppose that in the absence of a wave the fluid is stratified with a constant Brunt-Väisälä frequency N_0. We consider a harmonic internal inertial-gravity wave whose horizontal velocity field has the form

$$u_x=U_0\cos(\omega t-kx-\varkappa z),\quad u_y=U_0\frac{f}{\omega}\sin(\omega t-kx-\varkappa z).$$

For simplicity, we assume that $k\ll\varkappa$; then $\left|\frac{\partial u_x}{\partial z}\right|,\left|\frac{\partial u_y}{\partial z}\right|\gg\left|\frac{\partial w}{\partial z}\right|$, which makes it possible to take into account only the vertical turbulent flow of the horizontal component of the momentum

$$\langle u_x'w'\rangle=-L\sqrt{b}\frac{\partial u_x}{\partial z},\ \langle u_y'w'\rangle=-L\sqrt{b}\frac{\partial u_y}{\partial z}.$$

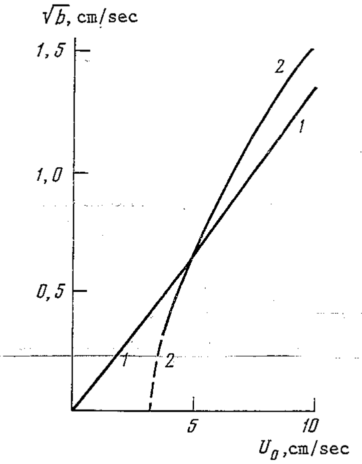

Fig. 4. Variation in the maximal value for $\sqrt{b}$ in the field of an internal inertial-gravity wave as a function of the amplitude for the horizontal velocity in the wave U_0: 1) using the model (13) and (14); 2) using the gradient hypothesis ($\varkappa_p = 0.1$).

The wave disturbance of the Brunt-Väisälä frequency has the form

$$N_{\sim}^2 = -N_0^2 U_0 k/\omega \sin(\omega t - kx - \varkappa z).$$

Neglecting turbulent diffusion, the equation for Π and b can be written as

$$\frac{db}{dt} = L\sqrt{b}U_0^2\varkappa^2\left[\cos^2(\omega t - kx - \varkappa z) + \frac{f^2}{\omega^2}\sin^2(\omega t - kx - \varkappa z) - \mathrm{Ri}_0\left(1 - \frac{kU_0}{\omega}\sin(\omega t - kx - \varkappa z)\right)\left(1 - \frac{3\Pi}{b}(1-R)\right)\right] - \frac{Cb^{3/2}}{L},$$

$$\frac{d\Pi}{dt} = L\sqrt{b}U_0^2\varkappa^2\mathrm{Ri}_0\left(1 - \frac{kU_0}{\omega}\sin(\omega t - kx - \varkappa z)\right)\left(1 - \frac{3\Pi}{b}(1-R)\right) - \frac{Cb^{1/2}\Pi}{L}. \quad (22)$$

Here $\mathrm{Ri}_0 = N_0^2/(U_0\varkappa)^2$.

In order to solve the system (22) numerically, values were chosen for the parameters that were close to those measured in the field experiments by Sanford [13]: $N = 5 \times 10^{-3}$ sec^{-1}, $\varkappa = 5 \times 10^{-2}$ m^{-1}, $\omega = 1.1 f$, $f = 7 \times 10^{-5}$ sec^{-1}, and $L = 1$ m, and U_0 varies in the range 1 - 10 cm/sec. The selection of the wave frequency close to the inertial one is due to the fact that it is precisely in this low-frequency range the velocity shear is greatest and, consequently, the mechanism for maintaining small-scale turbulence which is related to the direct transfer of velocity shear energy by turbulent fluctuations is most efficient. The variation in the maximal value for $\sqrt{b}$ in the field of an internal-gravity wave as a function of amplitude for the horizontal velocity in the field U_0 is shown in Fig. 4. This figure also contains for comparison the curve for the variation of $\sqrt{b}(U_0)$ obtained when the same parameters from the gradient hypotheses are used. We see the strong difference in these two functions in the region of small values for U_0.

Qualitative differences in the behavior of turbulence in the field of an internal inertial-gravity wave as compared with the results using gradient hypotheses also follow from Eqs. (13) and (14) used to close the Reynolds equations. Thus, it follows from the gradient hypotheses that for $f^2/\omega^2 < \varkappa_p \mathrm{Ri} < 1$ there is a "break" in the turbulence: at certain times its energy becomes zero, and a weak turbulent "injection" is needed for a new "burst." The break is due to the fact that for certain values of the wave phase the velocity shear is not sufficient to satisfy the condition for maintaining turbulence. The possibility of a break in the turbulence was noted in [12]. As has been shown above, the model (13) and (14) does not give a threshold value for the Richardson number, so therefore the break in the turbulence does not occur for any value of Ri. Actually, for large Ri ($\mathrm{Ri} \gg 1$, $\mathrm{Ri} \gg \omega/\varkappa U_0\sqrt{C}$) the turbulence energy satisfies an equation analogous to (21):

$$\frac{\partial b}{\partial t} = L\sqrt{b}\frac{3(1-R)}{4-3R}U_0^2\varkappa^2\left(\cos^2(\omega t - \varkappa z - kx) + \frac{f^2}{\omega^2}\sin^2(\omega t - kx - \varkappa z)\right) - \frac{b^{3/2}C}{L}, \quad (23)$$

in which Ri does not enter. If the wave period is long compared to the time for the increase in the turbulence (i.e., for $\omega/\varkappa U_0/\sqrt{C} \ll 1$), it is easy to write the quasistationary solution for Eq. (23):

$$b = \frac{L^2\varkappa^2U_0^2}{C(4-3R)}3(1-R)\left(\cos^2\omega t + \frac{f^2}{\omega^2}\sin^2\omega t\right). \quad (24)$$

We see that the turbulence energy given by Eq. (24) does not go to zero, and thus there is no break here.

In our view the conclusions of the theory developed here which takes into account the dual exchange between the kinetic and potential energies acts in favor of the explanation for the universal absence of turbulence within the ocean, which is related to the possibility of maintaining it by internal waves and shear flows at all Richardson numbers.

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